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Multivariate approximating averages

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Abstract

Averages on the sphere about x in R^d and on the rim of the cap about x in S^{d-1} and their iterates are shown to be smoother than f. Furthermore, their approximating properties satisfy a strong converse inequality of type A when dealing with multivariate approximation ($d \geqslant 2$ in case of R^d and $d \geqslant 3$ in case of S^{d-1}). These results are in contrast to the classical results on R or T for which the situation is completely different.

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1. Introduction

For a function $f(x)$ on $\boldsymbol{R}$ or $\boldsymbol{T}$, i.e. single variable, the average

$$A_t f(x) = \tfrac{1}{2}(f(x+t) + f(x-t)) \tag{1.1}$$

approximates $f(x)$ when $f(x) \in C(\boldsymbol{R})(C(\boldsymbol{T}))$ or $L_p(\boldsymbol{R})(L_p(\boldsymbol{T}))$, $1 \leqslant p < \infty$; and the rate of approximation relates to the modulus of smoothness $\omega^2(f,t)_p$ by

$$\sup_{0<u\leqslant t} ||f - A_u f||_p \equiv \omega^2(f,t)_p \approx \inf_g (||f-g||_p + t^2||g''||_p). \tag{1.2}$$

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It is well-known that $A_t f$ and its iterates $A_t^m f$ are not in general smoother than $f(x)$ and that the supremum on the left of (1.2) cannot be dropped.

We will show that some multivariate analogues which are widely used behave better and the behaviour improves with the dimension.

For $f(x)$, $x \in \boldsymbol{R}^d$ (or $\boldsymbol{T}^d$) we define

$$V_t(f,x) \equiv \frac{1}{m(t)} \int_{\sum_t = \{y:|x-y|=t\}} f(y)\, d\sigma, \quad V_t(1,x) = 1 \tag{1.3}$$

with $d\sigma$ being the Lebesgue measure on the sphere $\sum_t$.

For $f(x)$, $x \in S^{d-1} = \{x \in \boldsymbol{R}^d : |x| = 1\}$, we define

$$S_t(f,x) = \frac{1}{m(t)} \int_{\gamma_t = \{y: y\cdot x = \cos t\}} f(y)\, d\gamma, \quad S_t(1,x) = 1, \tag{1.4}$$

where $d\gamma$ is the Lebesgue measure on the rim of the cap of sphere γ_t.

It turns out that for $d \geqslant 2$

$$||V_t f(\cdot) - f(\cdot)||_{L_p(R^d)} \approx \inf_{g \in C^2(R^d)} (||f-g||_{L_p(R^d)} + t^2 ||\Delta g||_{L_p(R^d)}) \tag{1.5}$$

and for $d \geqslant 3$

$$||S_t f(\cdot) - f(\cdot)||_{L_p(S^{d-1})} \approx \inf_{g \in C^2(S^{d-1})} (||f-g||_{L_p(S^{d-1})} + t^2 ||\tilde{\Delta} g||_{L_p(S^{d-1})}), \tag{1.6}$$

where Δ and $\tilde{\Delta}$ are the Laplacian and the Laplace–Beltrami (the tangential component of the Laplacian on S^{d-1}), respectively.

We note that in (1.5) and in (1.6) proved here, there is no supremum on the left-hand side as is common in texts on the subject and as is necessary in the one-dimensional case. Equivalences (1.5) and (1.6) constitute strong converse inequalities of type A in the sense of [Di-Iv], but in one dimension (1.2) is a strong converse inequality of type D in that scale.

We show further that $V_t^m f$ and $S_t^m f$ are smoother than f, and the improvement depends on m, p (of L_p) and the dimension d. In fact, the higher the dimension, the bigger the improvement in smoothness of $S_t^m f$ and $V_t^m f$ over that of f. We note that in one dimension an improvement of smoothness of $A_t^m f$ is not the case for any p or m.

For the proof we will need and show that

$$||\Delta V_t^m(f)||_{L_p(R^d)} \leqslant \frac{\varepsilon(m)}{t^2} ||f||_{L_p(R^d)}, \quad \text{for } d \geqslant 2 \tag{1.7}$$

and

$$||\tilde{\Delta} S_t^m f||_{L_p(S^{d-1})} \leqslant \frac{\varepsilon(m)}{t^2} ||f||_{L_p(S^{d-1})} \quad \text{for } d \geqslant 3, \tag{1.8}$$

where $\varepsilon(m) = o(1)$, $m \to \infty$.

We note that for A_t^m we cannot guarantee the existence of a derivative, and hence there is no hope for an analogue of (1.7) and (1.8).

The technique we use relies on multipliers. The smoothness of functions on S^{d-1} will also be related to a new concept introduced recently in [Di-I].

We hope the present results will be helpful in other investigations, though we cannot identify the optimal improvement in smoothness except in the L_2 case.

2. Smoothness of $S_t^m f$ in $L_2(S^{d-1})$

We first show the smoothness of $S_t^m f$ in $L_2(S^{d-1})$, which is better than the result which we get for L_p in the next section (see also Section 7) when we substitute $p = 2$. The results in this section will serve as a model and encouragement for later sections and some of the estimates will be used later.

We have (see for instance [Li-Ni,Ru,Wa-Li, (12.4.6), p. 61])

$$S_\theta^m f = \sum_{k=0}^{\infty} \left(\frac{P_k^{(\lambda)}(\cos\theta)}{P_k^{(\lambda)}(1)} \right)^m P_k(f) \equiv \sum_{k=0}^{\infty} (Q_k^{(\lambda)}(\cos\theta))^m P_k(f), \tag{2.1}$$

where $P_k^{(\lambda)}(x)$ is the ultraspherical (Gegenbauer) polynomial of order k, $\lambda = \frac{d-2}{2}$, $P_k(f)$ is the projection of f on $H_k = \{\psi : \tilde{\Delta}\psi = -k(k+d-2)\psi\}$, $\tilde{\Delta} f(x) = \Delta f\left(\frac{x}{|x|}\right)$ (the Laplace–Beltrami operator) and Δ is the Laplacian. As both S_θ^m and $\tilde{\Delta}$ are multiplier operators, we have formally

$$-\tilde{\Delta} S_\theta^m f \sim \sum_{k=1}^{\infty} k(k+d-2) \left(\frac{P_k^{(\lambda)}(\cos\theta)}{P_k^{(\lambda)}(1)} \right)^m P_k(f). \tag{2.2}$$

Theorem 2.1. *For $f \in L_2(S^{d-1})$, $d \geqslant 3$, $\theta \leqslant \frac{\pi}{2}$, and $m \geqslant \frac{4}{d-2}$ we have*

$$\|\tilde{\Delta} S_\theta^m f\|_2 \leqslant \frac{C}{\theta^2} \|f\|_2, \tag{2.3}$$

with C independent of θ and f.

Proof. Using [Sz, (7.33.6)], we have (recalling that there $P_k^\lambda(1) = \binom{k+2\lambda-1}{k}$)

$$|Q_k^{(\lambda)}(\cos\theta)| \equiv \left| \frac{P_k^{(\lambda)}(\cos\theta)}{P_k^{(\lambda)}(1)} \right| \leqslant C_1 \min\{k^{-\lambda}\theta^{-\lambda}, 1\},$$

and hence (recalling $\lambda = \frac{d-2}{2}$) for $k\theta \geqslant 1$, $\theta \leqslant \frac{\pi}{2}$, we have

$$|\theta^2 k(k+d-2)(Q_k^{(\lambda)}(\cos\theta))^m| \leqslant C_2 (k\theta)^{2-\frac{d-2}{2}m}$$

which is bounded for $\frac{d-2}{2} m \geqslant 2$ or $m \geqslant \frac{4}{d-2}$. For $k\theta \leqslant 1$

$$|\theta^2 k(k+d-2)(Q_k^{(\lambda)}(\cos\theta))^m| \leqslant C_2,$$

which completes the proof. □

Corollary 2.2. *For* $f \in L_2(S^{d-1})$, $0 < \theta \leqslant \frac{\pi}{2}$, $d \geqslant 3$ and $m_1 \geqslant \frac{2}{d-2}$

$$|| \operatorname{grad}_{\tan} S_\theta^{m_1} f ||_2 \leqslant \frac{C}{\theta} ||f||_2, \tag{2.4}$$

with C *independent of* θ *and* f.

Proof. We have for smooth f

$$\begin{aligned} || \operatorname{grad}_{\tan} S_\theta^{m_1} f ||_2^2 &= \langle \operatorname{grad}_{\tan} S_\theta^{m_1} f, \operatorname{grad}_{\tan} S_\theta^{m_1} f \rangle \\ &= \langle -\tilde{\Delta} S_\theta^{2m_1} f, f \rangle \leqslant || \tilde{\Delta} S_\theta^{2m_1} f ||_2 ||f||_2 \\ &\leqslant \frac{C}{\theta^2} ||f||_2 ||f||_2, \end{aligned}$$

and as the last result is independent of the smoothness of f, (2.4) follows. □

We can also prove

Theorem 2.3. *For* $f \in L_2(S^{d-1})$, $d \geqslant 3$, $0 < \theta \leqslant \frac{\pi}{2}$, *and* m *big enough*

$$|| \tilde{\Delta} S_\theta^m f ||_2 \leqslant \frac{\varepsilon(m)}{\theta^2} ||f||_2, \tag{2.5}$$

where $\varepsilon(m) = o(1)$ *as* $m \to \infty$.

We need the following lemma, which will also be used later.

Lemma 2.4. *For* $n\theta \leqslant 1$, $n \geqslant 1$, $0 < \theta \leqslant \frac{\pi}{2}$ *and* $\lambda = \frac{d-2}{2}$

$$\left| \frac{P_n^{(\lambda)}(\cos\theta)}{P_n^{(\lambda)}(1)} \right| \equiv |Q_n^{(\lambda)}(\cos\theta)| \leqslant 1 - \frac{n^2}{d-1} \sin^2\frac{\theta}{2} \leqslant 1 - \frac{2}{\pi^2(d-1)} n^2\theta^2. \tag{2.6}$$

For $n\theta \geqslant 1$, $n \geqslant 1$, $0 < \theta \leqslant \frac{\pi}{2}$ *and* $\lambda = \frac{d-2}{2}$

$$|Q_n^{(\lambda)}(\cos\theta)| < \alpha < 1 \tag{2.7}$$

with α *independent of* n *and* θ (*but depending on the fact that* $n\theta \geqslant 1$ *and on* λ).

Proof. To show (2.6) we integrate by parts and use [Sz, (4.7.14), p. 82] with $\lambda = \frac{d-2}{2}$ to obtain for $n \geqslant 1$

$$\begin{aligned} 1 - Q_n^{(\lambda)}(\cos t) &= -\int_0^\theta \frac{d}{du} Q_n^{(\lambda)}(\cos u)\, du \\ &= \frac{n(n+d-2)}{d-1} \int_0^\theta Q_{n-1}^{(\lambda+1)}(\cos u) \sin u\, du \\ &= \frac{n(n+d-2)}{d-1} \left[\int_0^\theta \sin u\, du - \int_0^\theta \left(1 - Q_{n-1}^{(\lambda+1)}(\cos u)\right) \sin u\, du \right]. \end{aligned}$$

For $n=1$ $Q_n^{(\lambda)}(\cos\theta)=\cos\theta=1-2\sin^2\frac{\theta}{2}$. Otherwise, we use the mean value theorem to obtain

$$0\leqslant 1-Q_{n-1}^{(\lambda+1)}(\cos u)=(1-\cos u)\frac{(n-1)(n+d-1)}{d+1}Q_{n-2}^{(\lambda+2)}(\cos\xi)$$
$$\leqslant(1-\cos u)\frac{(n-1)(n+d-1)}{d+1},$$

and hence

$$\int_0^\theta\left(1-Q_{n-1}^{(\lambda+1)}(\cos u)\right)\sin u\,du\leqslant\frac{(n-1)(n+d-1)}{d+1}\frac{(1-\cos\theta)^2}{2}.$$

For $n\theta<1$ we have $\frac{(n-1)(n+d-1)}{d+1}(1-\cos\theta)\leqslant 1$, and hence

$$\begin{aligned}0&\leqslant 1-2\frac{n(n+d-2)}{d-1}\sin^2\frac{\theta}{2}\leqslant Q_n^{(\lambda)}(\cos\theta)\\&\leqslant 1-\frac{n(n+d-2)}{d-1}\sin^2\frac{\theta}{2}\\&\leqslant 1-\frac{n^2}{d-1}\sin^2\frac{\theta}{2}\leqslant 1-\frac{2}{(d-1)\pi^2}(n\theta)^2.\end{aligned}$$

For $n\theta\geqslant 1$, we use the identity [Sz, (4.10.3), p. 97]

$$\begin{aligned}Q_n^{(\lambda)}(\cos\theta)&=\left(\int_0^\pi\sin^{2\lambda-1}\varphi\,d\varphi\right)^{-1}\int_0^\pi(\cos\theta+i\sin\theta\cos\varphi)^n\sin^{2\lambda-1}\varphi\,d\varphi\\&=\left(\int_0^\pi\sin^{2\lambda-1}\varphi\,d\varphi\right)^{-1}\int_0^\pi\operatorname{Re}(\cos\theta+i\sin\theta\cos\varphi)^n\sin^{2\lambda-1}\varphi\,d\varphi.\end{aligned}\tag{2.8}$$

For $\theta_0\leqslant\theta\leqslant\pi-\theta_0$ we find δ such that

$$\int_0^\delta\sin^{2\lambda-1}\varphi\,d\varphi\Big/\int_0^\pi\sin^{2\lambda-1}\varphi\,d\varphi=\frac{1}{8},$$

and estimate for $n\geqslant 2$

$$\begin{aligned}|Q_n^{(\lambda)}(\cos\theta)|&\leqslant\tfrac{1}{4}+\tfrac{3}{4}|\cos\theta+i\sin\theta\cos\delta|^n\\&=\tfrac{1}{4}+\tfrac{3}{4}(1-\sin^2\theta\sin^2\delta)^{n/2}\\&\leqslant\tfrac{1}{4}+\tfrac{3}{4}(1-\sin^2\theta\sin^2\delta)\\&\leqslant 1-\tfrac{3}{4}\sin^2\theta_0\sin^2\delta,\end{aligned}$$

and for $n=1$ $Q_n^{\lambda}(\cos\theta)=\cos\theta\leqslant\cos\theta_0=1-2\sin^2\frac{\theta_0}{2}$, and hence (2.7) is valid for $\theta_0\leqslant\theta<\pi-\theta_0$ (not only for $\theta<\frac{\pi}{2}$) and α depending on θ_0 and λ (but not yet on $n\theta$). We choose θ_0 so that $\cos\theta_0\geqslant 0.9$ (for example) and note that Fejer showed [Sz, (6.6.6), p. 138] that the zeros of $P_n^{(\lambda)}(\cos\theta)$ satisfy

$$\frac{v-(1-\lambda)/2}{n+\lambda}\pi<\theta_v<\frac{v+\lambda-\frac{1}{2}}{n+2\lambda}\pi,$$

and hence the next extremum of $P_n^{(\lambda)}(\cos\theta)$ after $\theta=0$ is at a zero of $P_{n-1}^{(\lambda+1)}(\cos\theta)$ i.e. $\theta\leqslant\frac{1+\lambda+\frac{1}{2}}{n-1+2\lambda+2}\pi=\frac{1}{2}\frac{d+1}{n+d-1}\pi\leqslant\frac{1}{2}\frac{d+1}{n}\pi$.

As the $|P_n^{(\lambda)}(\cos\theta)|$ at the extrema are descending for $0\leqslant\theta\leqslant\frac{\pi}{2}$ (increasing in $x=\cos\theta$) [Sz, (1), pp. 168–169], we have to check the values only in the range $1\leqslant\theta n\leqslant\frac{(d+1)\pi}{2}$. Using (2.8), we note that for $n\leqslant\frac{(d+1)\pi}{2\theta}$ and $\tan^{-1}\frac{\sin\theta\cos\varphi}{\cos\theta}<\frac{\pi}{2n}$, $|\arg(\cos\theta+i\sin\theta\cos\varphi)^n|<\frac{\pi}{2}$, and hence $\mathrm{Re}(\cos\theta+i\sin\theta\cos\varphi)^n>0$. Clearly, $\tan^{-1}\left(\frac{\sin\theta\cos\varphi}{0.9}\right)<\tan^{-1}\frac{\theta\cos\varphi}{0.9}<\frac{\theta\cos\varphi}{0.9}$ and hence for $\frac{\theta\cos\varphi}{0.9}<\frac{\pi}{2n}$ or $\cos\varphi\leqslant 0.9\frac{\pi}{2n\theta}\leqslant 0.9\frac{\pi}{(d+1)\pi}=0.9\frac{1}{d+1}$. We choose φ_0 such that $\cos\varphi_0=0.9\frac{1}{d+1}$ and as

$$\left(\int_{\varphi_0}^{\pi-\varphi_0}\sin^{2\lambda-1}\varphi\,d\varphi\right)\Big/\int_0^{\pi}\sin^{2\lambda-1}\varphi\,d\varphi\geqslant\beta>0.$$

with β depending on λ only, we have for $\theta n\leqslant\frac{(d+1)\pi}{2}$, using (2.8)

$$Q_n^{(\lambda)}(\cos\theta)\geqslant-(1-\beta)\max|\cos\theta+i\sin\theta\sin\varphi|^n\geqslant-1+\beta,$$

and hence $Q_n^{(\lambda)}(\cos\theta)\geqslant-1+\beta$ at the first extremum after $\theta=0$. Therefore, for $n\theta\geqslant1$, $\theta\leqslant\theta_0<\frac{\pi}{2}$

$$|Q_n^{(\lambda)}(\cos\theta)|\leqslant\max\left(1-\frac{2}{\pi^2}\frac{1}{d-1},1-\beta\right)=\alpha<1. \qquad\square$$

Proof of Theorem 2.3. For $n\theta\leqslant1$ we use (2.6) of Lemma 2.4 to estimate the coefficients

$$\theta^2n(n+d-2)\left(1-\frac{2}{\pi^2(d-1)}(n\theta)^2\right)^m\leqslant(d-1)\theta^2n^2\left(1-\frac{2}{\pi^2(d-1)}(n\theta)^2\right)^m$$
$$\leqslant C\frac{1}{m}=o(1),\quad m\to\infty.$$

For $n\theta\geqslant1$ we write $m=m_1+m_2$ where $m_1\geqslant\frac{4}{d-2}$ from Theorem 2.1, and use (2.7) of Lemma 2.4 to get $\alpha^{m_2}=o(1)$, $m_2\to\infty$, which concludes the proof. $\square$

3. Smoothness of $S_\theta^m f$ for $f\in L_p(S^{d-1})$, $1\leqslant p\leqslant\infty$

In this section we exhibit the smoothness of $S_\theta^m f$ for some power m.

Theorem 3.1. *For $f\in L_p(S^{d-1})$, $1\leqslant p\leqslant\infty$, $d\geqslant3$ and $m>\frac{2([\frac{d}{2}]+3)}{d-2}$ we have*

$$||\tilde{\Delta}S_\theta^m f||_p\leqslant C\max\left(\frac{1}{\theta^2},\frac{1}{(\pi-\theta)^2}\right)||f||_p. \tag{3.1}$$

Remark 3.2. Theorem 3.1 does not contain Theorem 2.1 as the condition on m is harsher. In Section 7 we will improve the condition on m to be close to that in Theorem 2.1 for p close to 2.

We need the following lemma:

Lemma 3.2. *For integers j, k and m and for $0<\theta\leqslant\frac{\pi}{2}$ we have*

$$\left|\Delta^j\left(Q_k^{(\lambda)}(\cos\theta)\right)^m\right|\leqslant\begin{cases} C\theta^j/(k\theta)^{m\lambda} & \text{for } k\theta\geqslant 1,\\ C\theta^j & \text{for } k\theta\leqslant 1,\end{cases} \tag{3.2}$$

where C is independent of k and θ but may depend on j, λ and m, and where $\Delta a_k \equiv \vec{\Delta} a_k \equiv a_{k+1}-a_k$, $\Delta^j a_k \equiv \Delta(\Delta^{j-1}a_k)$.

Proof. Using Leibnitz's theorem for differences, we only have to show (3.2) for $m=1$ and all j. This result follows from classical estimates and was essentially demonstrated in [Da-Wa-Yu, (21)]. For $j=0$, $m=1$ (3.2) is in [Sz, (7.33.6), p. 170]. For α, β one uses [Sz, (4.5.4), p. 71] to obtain for $Q_k^{(\alpha,\beta)}(x)\equiv P_k^{(\alpha,\beta)}(x)/P_k^{(\alpha,\beta)}(1)$

$$Q_{k+1}^{(\alpha,\beta)}(x)-Q_k^{(\alpha,\beta)}(x)=-(1-x)\frac{(2k+\alpha+\beta+2)}{2(\alpha+1)}Q_k^{(\alpha+1,\beta)}(x)$$

and by induction

$$\Delta^j Q_k^{(\alpha,\beta)}(x)=\sum_{\ell\geqslant j/2}^{j}(1-x)^{\ell}P_\ell(k)Q_k^{(\alpha+\ell,\beta)}(x),$$

where $P_\ell(k)$ are polynomials of degree $2\ell-j$ in k. From the above, using $\alpha=\lambda-\frac{1}{2}$ with [Sz, (8.21.18), p. 196] for $k\theta\geqslant 1$ and $|Q_k^{(\alpha+j,\beta)}(x)|\leqslant 1$ for $k\theta<1$, we complete the proof of (3.2). $\square$

Remark 3.3. For $\theta\geqslant\frac{\pi}{2}$

$$S_\theta f(x)=S_{\pi-\theta}f(x_0)$$

with x_0 the antipodal of x, that is the other point of intersection of S^{d-1} with the line connecting x with the center. Observe that

$$S_\theta^m f(x)=S_{\pi-\theta}^m f(x) \quad \text{for even } m$$

and

$$S_\theta^m f(x)=S_{\pi-\theta}^m f(x_0) \quad \text{for odd } m.$$

Proof of Theorem 3.1. In view of Remark 3.3 we have to prove our result only for $0<\theta\leqslant\frac{\pi}{2}$. It was shown in [Bo-Cl] that the Cesàro summability of order r satisfies $||\sigma_n^r f||_{L_p(S^d)}\leqslant C||f||_{L_p(S^d)}$, for any $r>\frac{d-2}{2}$ (and actually for $r>(d-2)|\frac{1}{2}-\frac{1}{p}|$). We will use the well-known result relating

$$-\tilde{\Delta} S_\theta^m f \sim \sum_{k=1}^{\infty} k(k+d-2)(Q_k^{(\lambda)}(\cos\theta))^m P_k(f) \tag{3.3}$$

to

$$-\theta^2 \tilde{\Delta} S_\theta^m f = \theta^2 \sum_{k=1}^{\infty} \Delta^{r+1}\left\{k(k+d-2)\left(Q_k^{(\lambda)}(\cos\theta)\right)^m\right\}\binom{k+r}{r}\sigma_k^r(f)$$

(which follows from the identity $P_k(f) = \overleftarrow{\Delta}^{r+1}\binom{k+r}{r}\sigma_n^r(f,x)$, using summation by parts) if the latter is convergent termwise in norm. It is therefore sufficient to show that

$$\theta^2 \sum_{k=1}^{\infty} \left|\Delta^{r+1}\left\{k(k+d-2)\left(Q_k^{(\lambda)}(\cos\theta)\right)^m\right\}\right|\binom{k+r}{r} \leqslant C_1$$

or

$$S = \theta^2 \sum_{k=1}^{\infty} \left|\Delta^{r+1-i}\left(Q_k^{(\lambda)}(\cos\theta)\right)^m\right| k^{r+2-i} \leqslant C_2$$

for $i = 0, 1, 2$. We separate the sum S by $S = S_1 + S_2$ where S_1 sums on k satisfying $k\theta \geqslant 1$ and S_2 when $k\theta \leqslant 1$.

Using Lemma 3.2, we have

$$\begin{aligned} S_1 &\leqslant C_3\theta^2 \sum_{k\theta\geqslant 1} \frac{1}{(k\theta)^{\lambda m}}\theta^{r+1-i}k^{r+2-i} \\ &\leqslant C_4 \theta^{2-\lambda m + r + 1 - i} \sum_{k\theta \geqslant 1} k^{r+2-i-\lambda m}. \end{aligned}$$

When $r + 2 - i - \lambda m < -1$, that is, $\frac{r+3}{\lambda} < m$ or $\frac{2(r+3)}{d-2} < m$, S_1 is bounded.

To estimate S_2 we write

$$S_2 \leqslant C_5 \theta^2 \sum_{k\theta \leqslant 1} \left(\frac{2}{k}\right)^{r+1-i} k^{r+2-i} \leqslant C_6 \theta^2 \sum_{k\theta\leqslant 1} k \leqslant C_6. \qquad \square$$

Perhaps as an intuitive incentive to reduce m (but certainly not a proof) we observe that the same method used to prove Theorem 3.1 yields:

Theorem 3.4. *For $f \in L_p(S^{d-1})$ $d \geqslant 3$ and $m > \frac{2(r+2\ell+1)}{(d-2)\ell}$ we have*

$$||\tilde{\Delta}^\ell S_\theta^{\ell m} f||_p \leqslant C \max\left(\frac{1}{\theta^{2\ell}}, \frac{1}{(\pi-\theta)^{2\ell}}\right)||f||_p. \tag{3.4}$$

We will not elaborate on the proof of Theorem 3.4 as it follows almost word for word that of Theorem 3.1.

Remark 3.5. For $1<p<\infty$ and $f\in L_p(S^{d-1})$ we could have now deduced for $0<\theta\leqslant\frac{\pi}{2}$

$$\|\tilde{\Delta}S_\theta^m f\|_p\leqslant\frac{\varepsilon(m)}{\theta^2}\|f\|_p$$

using Theorems 2.1, 3.1 and the Riesz–Thorin interpolation theorem. However, the result is valid for $p=1$ and ∞ as well, as will be shown (with substantial additional work) in Section 4.

4. Estimate of $S_\theta^m f$ for high power m

The result of this section is the following theorem.

Theorem 4.1. *For $f\in L_p(S^{d-1})$, $1\leqslant p\leqslant\infty$, $d\geqslant 3$ and m big enough*

$$\|\tilde{\Delta}S_\theta^m f\|_p\leqslant\varepsilon(m)\max\left(\frac{1}{\theta^2},\frac{1}{(\pi-\theta)^2}\right)\|f\|_p, \tag{4.1}$$

where $\varepsilon(m)\to 0$ as $m\to\infty$.

Proof. It is sufficient in view of Remark 3.3 to show (4.1) for $0<\theta\leqslant\frac{\pi}{2}$. We write, following the proof of Theorem 3.1,

$$-\theta^2\tilde{\Delta}S_\theta^m f=\theta^2\sum_{k=1}^{\infty}\Delta^{r+1}\Big((k(k+d-2))\Big(Q_k^{(\lambda)}(\cos\theta)\Big)^m\Big)\binom{k+r}{r}\sigma_k^r(f),$$

and we have to show

$$S=\theta^2\sum_{k=1}^{\infty}\left|\Delta^{r+1}(k(k+d-2))\Big(Q_k^{(\lambda)}(\cos\theta)\Big)^m\right|k^r\leqslant\varepsilon(m),\quad m\to\infty.$$

We set

$$S\leqslant C(d)\theta^2\sum_{i=0}^{2}\sum_{k=1}^{\infty}\left|\Delta^{r+1-i}Q_k^{(\lambda)}(\cos\theta)^m\right|k^{r+2-i}\equiv\sum_{i=0}^{2}S(i).$$

We estimate $S(0)$, and estimates for $S(1)$ and $S(2)$ are almost identical. We write now, choosing $\ell>\frac{r+3}{\lambda}\geqslant\frac{2(d+5)}{d-2}$ and $\ell\geqslant r+1$ say $\ell=\max\left(\left[\frac{2(d+5)}{d-2}\right]+1,\left[\frac{d-2}{2}\right]+2\right)$ (which is perhaps lavish),

$$\begin{aligned}S(0)\leqslant\ & C(d)\theta^2\left[\sum_{1\leqslant k\leqslant\frac{1}{\theta}}+\sum_{k>\frac{1}{\theta}}\right]\left\{m^{r+1}A_{k,\ell}(\theta)\left(\max_{0\leqslant s\leqslant r+1}\left|Q_{k+s}^{(\lambda)}(\cos\theta)\right|\right)^{m-\ell}k^{r+2}\right\}\\ \equiv\ & S_1+S_2,\end{aligned}$$

where

$$A_{k,\ell}(\theta) = \max\left\{ \prod_{1\leqslant i\leqslant \ell} \left| \Delta^{j_i} Q^{(\lambda)}_{k+s}(\cos\theta)\right|; 0\leqslant j_i\leqslant r+1, \sum_{i=1}^{\ell} j_i = r+1, 0\leqslant s\leqslant r \right\}.$$

For S_1 we note that, using the estimate in Lemma 3.2, we have

$$S_1 \leqslant C\theta^2 \sum_{1\leqslant k\leqslant \frac{1}{\theta}} m^{r+1}\theta^{r+1}(B_k(\theta))^{m-\ell}k^{r+2}$$
$$\leqslant C\theta^2 \sum_{1\leqslant k\leqslant \frac{1}{\theta}} km^{r+1}(k\theta)^{r+1}(B_k(\theta))^{m-\ell},$$

where $B_k(\theta) = \max_{0\leqslant s\leqslant r+1} |Q^{(\lambda)}_{k+s}(\cos\theta)|$. Using (2.6) of Lemma 2.4, we have for $m>2\ell$ (we note that using (2.7) the range in which (2.6) is valid can be extended to include $k+s\leqslant k+r+1$)

$$m^{r+1}(k\theta)^{r+2}(B_k(\theta))^{m-\ell} \approx \tfrac{1}{m},$$

and hence

$$S_1 \approx \frac{1}{m}\theta \sum_{1\leqslant k\leqslant \frac{1}{\theta}} \approx \frac{1}{m}.$$

To estimate S_2 we use the estimate in Lemma 3.2 and write

$$S_2 \leqslant C\theta^2 \sum_{k>\frac{1}{\theta}} [m^{r+1}B_k(\theta)^{m-\ell}]k^{r+2}\frac{\theta^{r+1}}{(k\theta)^{\lambda\ell}}.$$

Using (2.7) of Lemma 2.4,

$$m^{r+1}Q^{(\lambda)}_{k+s}(\cos\theta)^{m-\ell} = o(1) \text{ as } m\to\infty \text{ and } k\theta>1 \text{ and } s\leqslant r+1$$

and hence for ℓ chosen earlier $S_2 = o(1)$, $m\to\infty$ in a manner that is dependent only on λ (r is fixed by λ as well). $\square$

5. The equivalence result and other corollaries

As a corollary of the results in Sections 3 and 4 we will obtain the equivalence result or strong converse inequality of type A (in the terminology of [Di-Iv]).

Theorem 5.1. *For* $f\in L_p(S^{d-1})$, $1\leqslant p\leqslant\infty$, $d\geqslant 3$ *and* $0<\theta\leqslant\frac{\pi}{2}$

$$\|S_\theta f - f\|_p \approx \inf_g(\|f-g\|_p + \theta^2\|\tilde{\Delta}g\|_p) \equiv K(f,\theta^2)_p, \tag{5.1}$$

where the infimum is taken on sufficiently smooth g.

Remark 5.2. The common form of the relationship between $S_\theta f - f$ and $K(f,\theta^2)_p$ is

$$\sup_{\theta \leqslant t} ||S_\theta f - f||_p \approx \inf_g (||f-g||_p + t^2||\tilde{\Delta} g||_p), \tag{5.2}$$

and it is the dropping of the supremum on the left of (5.2) which defined smoothness in all previous papers on the subject (scores of them) that is a highlight in this paper.

To prove (5.1) we follow the method of [Di-Iv, Section 4] but for that we need in addition to Theorem 4.1 the following improved Voronovskaja-type result for S_θ.

Lemma 5.3. *For g such that $\tilde{\Delta}^i g \in L_p(S^{d-1})$, $1 \leqslant p < \infty$, for $i = 0,1,2$ and $\tilde{\Delta}^i g \in C$ for $p = \infty$ and for θ satisfying $0 < \theta \leqslant \frac{\pi}{2}$*

$$||S_\theta g - g - \alpha(\theta)\tilde{\Delta} g||_p \leqslant C\theta^4 ||\tilde{\Delta}^2 g||_p \tag{5.3}$$

with $0 < A\theta^2 \leqslant \alpha(\theta) \leqslant B\theta^2$ and A, B, C independent of θ and g.

Proof. Using Theorem 2.1 of [Di-RuII] for $g \in C^2(S^{d-1})$,

$$S_\theta(g,x) - g(x) = \frac{\Gamma\left(\frac{d-1}{2}\right)}{2\pi^{(d-1)/2}} \int_0^\theta \frac{dt}{(\sin t)^{d-2}} \int_{\sigma_x(t)} \tilde{\Delta} g(y)\, d\sigma(y), \tag{5.4}$$

where $\sigma_x(t) = \{y; |y| = 1,\ \cos t \leqslant x \cdot y \leqslant 1\}$. Repeating this process as was done in Lemma 4.2 of [Di-RuII], we obtain (5.3) for $g \in C^4(S^{d-1})$. Using density of $C^4(S^{d-1})$ in the space for which $\tilde{\Delta}^i g$ for $i = 0,1,2$ are in L_p, $1 \leqslant p < \infty$ and in C for $p = \infty$, we complete the proof. □

Proof of Theorem 5.1. We follow [Di-Iv, Section 4]. For g satisfying $g, \tilde{\Delta} g \in L_p$, or g, $\tilde{\Delta} g \in C$ (or $p = \infty$) (5.4) implies

$$||S_\theta g - g||_p \leqslant C\theta^2 ||\tilde{\Delta} g||_p,$$

and as S_θ is a contraction on L_p, $1 \leqslant p \leqslant \infty$, we have the known direct result

$$||S_\theta f - f||_p \leqslant CK(f,\theta^2)_p.$$

To prove the converse (strong converse inequality of type A in the terminology of [Di-Iv]), we use for g of (5.3) $S_\theta^m f$ with m to be chosen. Clearly,

$$||f-g||_p = ||f - S_\theta^m f||_p \leqslant m||f - S_\theta f||_p.$$

As we established that $S_\theta^m f$ and $\tilde{\Delta} S_\theta^m f$ are in L_p for m big enough, it remains to show that for some m

$$\theta^2 ||\tilde{\Delta} S_\theta^m f||_p \leqslant C||f - S_\theta f||_p. \tag{5.5}$$

To prove (5.5) we first observe that using Theorem 3.1 with $m > \frac{4\left(\left[\frac{d}{2}\right]+3\right)}{d-2}$, $\tilde{\Delta} S_\theta^m f$ and $\tilde{\Delta}^2 S_\theta^m f$ are in L_p $1 \leqslant p \leqslant \infty$ if f is. Furthermore, if $m > \frac{6\left(\left[\frac{d}{2}\right]+3\right)}{d-2}$ and $f \in L_\infty(S^{d-1})$,

$\tilde{\Delta}^3 f \in L_\infty(S^{d-1})$, and this implies that $\tilde{\Delta}^2 f$ is continuous. We now choose $m = \frac{4\left(\left[\frac{d}{2}\right]+3\right)}{d-2} + m_1$ where m_1 $\left(m_1 \geqslant \frac{2\left[\frac{d}{2}\right]+3}{d-2}\right)$ is such that for all $f \in L_p$

$$\|\tilde{\Delta} S_\theta^{m_1} f\|_p \leqslant \frac{A}{2C} \frac{1}{\theta^2} \|f\|_p \tag{5.6}$$

with A and C of Lemma 5.3. The possibility of such a choice is guaranteed by Theorem 4.1. We use Lemma 5.3 on $g = S_\theta^m f$ and obtain

$$\begin{aligned}\|\alpha(\theta)\tilde{\Delta} S_\theta^m f\|_p &\leqslant \|S_\theta^m f - f\|_p + C\theta^4 \|\tilde{\Delta}^2 S_\theta^m f\| \\ &\leqslant m\|S_\theta f - f\|_p + \frac{A}{2}\theta^2 \|\tilde{\Delta} S_\theta^{m-m_1} f\|_p \\ &\leqslant m\|S_\theta f - f\|_p + \frac{A}{2}\theta^2 \|\tilde{\Delta} S_\theta^m f\|_p + \frac{A\theta^2}{2} \|\tilde{\Delta} S_\theta^{m-m_1}(S_\theta^{m_1} - I) f\|_p \\ &\leqslant m\|S_\theta f - f\| + \frac{A}{2} C_1 \|(S_\theta^{m_1} - I) f\| + \frac{A}{2}\theta^2 \|\tilde{\Delta} S_\theta^m f\|_p,\end{aligned}$$

with C_1, the constant from Theorem 3.1. Use of the triangle inequality will complete the proof. □

We also have as a corollary of the above and Theorem 10.4 of [Di-Iv]:

Corollary 5.4. *For* $f \in L_p(S^{d-1})$, $1 \leqslant p \leqslant \infty$, $d \geqslant 3$, $0 < \theta \leqslant \frac{\pi}{2}$ and integer ℓ

$$\|(S_\theta - I)^\ell f\|_p \approx \inf_g (\|f - g\|_p + \theta^{2\ell} \|\tilde{\Delta}^\ell g\|_p) \equiv K_r(f, \tilde{\Delta}, \theta^{2r})_p. \tag{5.7}$$

Proof. We just have to verify that S_θ satisfies the conditions (for Q_d) in Theorem 10.4 of [Di-Iv], which it does. □

6. The result on R^d or T^d

We define $V_t f$ for f on $\boldsymbol{R}^d$ or $\boldsymbol{T}^d$ by

$$V_t(f, x) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(x + ty)\, d\sigma(y), \tag{6.1}$$

where S^{d-1} is the unit sphere in $\boldsymbol{R}^d$, $|S^{d-1}|$ is its measure, and the integration $d\sigma(y)$ is on the sphere. It is well-known that V_t is a contraction on $L_p(\boldsymbol{R}^d)$ or $L_p(\boldsymbol{T}^d)$, $1 \leqslant p \leqslant \infty$. We will deal with $\boldsymbol{R}^d$ and show how to copy the results to $\boldsymbol{T}^d$.

Using [St-We, p. 154], we have for $x \in \boldsymbol{R}^d$ and $v = \frac{d-2}{2}$

$$\widehat{\Delta V_t^m} f(x) = (-4\pi^2 |x|^2) \cdot \left(\Gamma\left(\frac{d}{2}\right) \frac{J_v(2\pi t|x|)}{(\pi t|x|)^v}\right)^m \hat{f}(x), \tag{6.2}$$

where

$$\hat{f}(x) = \int_{R^d} f(y) e^{-2\pi i x y}\, dy.$$

We can now prove the following result about the smoothness of $V_t f$.

Theorem 6.1. *For $f \in L_2(\boldsymbol{R}^d)$, $d \geqslant 2$ and $m \geqslant \frac{4}{d-1}$*

$$\|\Delta V_t^m f\|_2 \leqslant \frac{C}{t^2} \|f\|_2. \tag{6.3}$$

Proof. Using (6.2), it is sufficient to show that for $\nu = \frac{d-2}{2}$

$$t^2|x|^2 \left| \frac{\Gamma\left(\frac{d}{2}\right) J_\nu(2\pi t|x|)}{(\pi t|x|)^\nu} \right|^m = I(m, \nu, t|x|)$$

is bounded. For $t|x| \leqslant 1$ $I(m, \nu, t|x|)$ is bounded by 1 which follows from the definition of $J_\nu(u)$ [St-We, p. 154], as

$$\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} (1 - s^2)^{(d-3)/2}\, ds = 1.$$

For $t|x| \geqslant 1$ we use Lemma 3.11 in [St-We, p. 158] to obtain

$$\frac{|J_\nu(2\pi t|x|)|}{(\pi t|x|)^\nu} \leqslant \frac{C}{(2\pi t|x|)^{\nu + \frac{1}{2}}}$$

and $I(m, \nu, t|x|)$ is bounded if $2 - m\left(\nu + \frac{1}{2}\right) \leqslant 0$ or $m \geqslant \frac{4}{d-1}$. $\square$

Theorem 6.2. *For $f \in L_p(\boldsymbol{R}^d)$, $d \geqslant 2$ and $m > \frac{2(d+2)}{d-1}$*

$$\|\Delta V_t^m f\|_p \leqslant \tfrac{C}{t^2} \|f\|_p. \tag{6.4}$$

Proof. A simple change of variables implies that it is sufficient to prove (6.4) for $t = 1$. Furthermore, it is sufficient to prove (6.4) for $p = 1$, which implies (6.4) for $p = \infty$ by duality, and for $1 < p < \infty$ by the Riesz–Thorin Theorem. To prove it now (for $L_1(\boldsymbol{R}^d)$) it is sufficient to show that

$$|x|^2 \left(\frac{\Gamma\left(\frac{d}{2}\right) J_\nu(2\pi|x|)}{(\pi|x|)^\nu} \right)^m \equiv I(m, \nu, |x|)$$

is in $L_1(\boldsymbol{R}^d)$ and is a Fourier transform of an element in L_1 for the given m. For any m and $|x|\leqslant 1$, $I(m,v,|x|)\leqslant 1$. To show that $I(m,v,|x|)$ is in L_1 it suffices to show for $|x|\geqslant 1$ $|I(m,v,|x|)|\leqslant\frac{C}{|x|^{d+\alpha}}$ with $\alpha>0$. Following Lemma 3.11 of [St-We, p. 158], this estimate is achieved for $d+2<m\left(\frac{d-1}{2}\right)$ or for $m>\frac{2(d+2)}{d-1}$. Therefore, $I(m,v,|x|)$ is the Fourier transform of a bounded continuous function. To show that it is a Fourier transform of a L_1 function it is sufficient to show that $d+1$ derivatives of $I(m,v,|x|)$ are bounded in L_1. Using the recursion relation

$$\frac{d}{du}(u^{-v}J_v(u)) = -u^{-v}J_{v+1}(u)$$

(see [St-We, p. 153]) and the split used above, we obtain that the derivatives of $I(m,v,|x|)$ are in L_1 and hence its inverse Fourier transform is in L_1. (Actually, it is continuous and of support in $|u|\leqslant m$.) We have $||\mathscr{F}^{-1}(I(m,v,|x|))||_1\leqslant C$, and hence our result for L_1 and $t=1$ from which (6.4) follows. □

Remark 6.3. We do not know how near the power m required in Theorem 6.2 is to the optimal for the space L_1 or L_∞, but see also Theorem 7.7.

We will now give the following crucial estimate.

Theorem 6.4. *For $f\in L_p(\boldsymbol{R}^d)$, $d\geqslant 2$ we have*

$$||\Delta V_t^m f||_p \leqslant \frac{\varepsilon(m)}{t^2}||f||_p \tag{6.5}$$

with $\varepsilon(m)\to 0$ as $m\to\infty$.

Proof. As discussed in the proof of Theorem 6.2, it is sufficient to show that

$$\left\|\mathscr{F}^{-1}\left(|x|^2\left(\frac{\Gamma\left(\frac{d}{2}\right)J_v(2\pi|x|)}{(\pi|x|)^v}\right)^m\right)\right\|_{L_1(R^d)}\to 0 \quad \text{as } m\to\infty.$$

To show the latter, we have to show that

$$\left\|D^\mu\left\{|x|^2\left(\frac{\Gamma\left(\frac{d}{2}\right)J_v(2\pi|x|)}{(\pi|x|)^v}\right)^m\right\}\right\|_1 \to 0 \quad \text{as } m\to\infty$$

for the derivatives D^μ where $|\mu|\leqslant d+1$ (including $\mu=0$ and $v=\frac{d-2}{2}$). For $|\mu|=\ell$, $\ell\leqslant d+1$ set the domain

$$D_\alpha = \left\{x : |x|\leqslant\frac{1}{m^\alpha}\right\}$$

with α to be chosen ($\alpha=\frac{1}{4}\left(\frac{2d+\frac{5}{2}}{d+\frac{3}{2}}\right)$ will do).

We now write for $\frac{d+1}{2d+3}<\alpha<\frac{1}{2}$

$$\left\| D^{\mu}\left\{ |x|^2 \left(\frac{\Gamma\left(\frac{d}{2}\right) J_v(2\pi|x|)}{(\pi|x|)^v} \right)^m \right\} \right\|_{L_1(D_\alpha)} \leqslant \frac{C}{m^{(d+2)\alpha}} m^{|\mu|} \left(\frac{1}{m^\alpha} \right)^{|\mu|} = o(1), \quad m\to\infty, \quad |\mu|\leqslant d+1,$$

using the recursion relation $\frac{d}{dt}(t^{-v}J_v(t)) = -t^{-v}J_{v+1}(t)$ and the trivial estimate $J_{v+1}(t) = O(t^{v+1})$, $t\to 0$. For $\frac{1}{m^\alpha}\leqslant|x|\leqslant\frac{1}{3}$ we use the estimate ($v=\frac{d-2}{2}$)

$$\begin{aligned} I &= m^{|\mu|} \left| \frac{\Gamma\left(\frac{d}{2}\right)}{(\pi|x|)^v} J_v(2\pi|x|) \right|^{m-\ell} \\ &\leqslant m^{|\mu|} \left| \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} e^{i2\pi|x|s}(1-s^2)^{\frac{d-3}{2}}\, ds \right|^{m-\ell} \\ &\leqslant m^{|\mu|} \left| \tfrac{1}{2}(\cos 2\pi m^{-\alpha}\delta + 1) \right|^{m-\ell}, \end{aligned}$$

where $\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-\delta}^{\delta}(1-s^2)^{(d-3)/2}\, ds = \frac{1}{2}$ (recall $\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1}(1-s^2)^{(d-3)/2}\, ds = 1$).

Therefore,

$$I\leqslant m^{|\mu|}(1-\sin^2(\pi m^{-\alpha}\delta))^{m-\ell}.$$

We choose ℓ to be any fixed number $\ell>\frac{2(d+2)}{d-1}$ from Theorem 6.2. (For example $\ell = \left[\frac{2(d+2)}{d-1}\right]+1$.)

We can now write, as $\sin^2 u\geqslant\frac{2^2u^2}{\pi^2}$ for $u\leqslant\frac{\pi}{2}$,

$$I\leqslant m^{|\mu|}(1-4\delta^2 m^{-2\alpha})^{m-\ell}.$$

If $\alpha<\frac{1}{2}$, $I=o(1)$ as $m\to\infty$.

To complete the estimate, it is now sufficient to observe that

$$\tilde{I}(x,v) = \left| \frac{\Gamma\left(\frac{d}{2}\right)}{(\pi|x|)^v} J_v(2\pi|x|) \right| \leqslant\beta<1 \quad \text{for } |x|\geqslant\tfrac{1}{3}, \quad v=\tfrac{d-2}{2}.$$

For $|x|\geqslant A$ with A big enough that estimate follows from [St-We, Lemma 3.11, p. 158]. For $\frac{1}{3}\leqslant|x|\leqslant A$ $|\tilde{I}(x,v)|$ achieves a maximum at x_0 as it is continuous, but for any $x_0\neq 0$ $\beta=|\tilde{I}(x_0,v)|<1$ using the representation in [St-We, pp. 153–154].

We now choose $\frac{d+1}{2d+3}<\alpha<\frac{1}{2}$, and clearly $\alpha=\frac{1}{4}\left(\frac{2d+\frac{5}{2}}{d+\frac{3}{2}}\right)$ will do. This completes the proof. □

Let us now transfer the theorems of this section to $L_p(\boldsymbol{T}^d)$.

Theorem 6.5. *In Theorems* 6.2 *and* 6.4 $L_p(\boldsymbol{R}^d)$ *can be replaced by* $L_p(\boldsymbol{T}^d)$. *In Theorem* 6.1 $L_2(\boldsymbol{R}^d)$ *can be replaced by* $L_2(\boldsymbol{T}^d)$.

Proof. As Theorems 6.2 and 6.4 are valid for $L_\infty(\boldsymbol{R}^d)$, they are valid for $L_\infty(\boldsymbol{T}^d)$ since $f \in L_\infty(\boldsymbol{T}^d)$ can be extended to $f \in L_\infty(\boldsymbol{R}^d)$ by periodicity without changing the norm. Being valid for $L_\infty(\boldsymbol{T}^d)$, Theorems 6.2 and 6.4 follow using duality for $L_1(\boldsymbol{T}^d)$, and using the Riesz–Thorin theorem for $L_p(\boldsymbol{T}^d)$, $1 < p < \infty$. For L_2 the result (Theorem 6.1) follows as we have essentially the same multipliers. □

Corollary 6.6. *For* $f \in L_p(\boldsymbol{R}^d)$ *or* $f \in L_p(\boldsymbol{T}^d)$, $1 \leqslant p \leqslant \infty$ *we have*

$$\|V_t f - f\|_p \approx \inf_g (\|f - g\|_p + t^2 \|\Delta g\|_p). \tag{6.6}$$

Proof. The proof follows that in Section 5 where we use iterations of [Di-RuI, Theorem 2.1] as seen in the Voronovskaja-type result [Di-RuI, (3.4)] in combination with Theorems 6.2 and 6.4. As the technique is almost exactly the same, we omit the details. □

7. Smoothness of $S_\theta f$ and $V_t f$

It is evident that the results on smoothness in L_2, that is, Theorem 2.1, Corollary 2.2 and Theorem 6.1 are optimal. For L_p, $p \neq 2$ the results we have are not optimal. However, we will show below how for p near 2 we can obtain results close to those for $p = 2$. It follows from Theorem 3.1 that for $m > \frac{2\left(\left[\frac{d}{2}\right]+3\right)}{d-2}$, $\tilde{\Delta} S_\theta^m f$ exists in L_1. We will show that $S_\theta f$ does not satisfy any smoothness property in L_1. Similar results are valid for $V_t f$.

We define for $K_r(f, \tilde{\Delta}, t^{2r})_p$ given in (5.7)

$$\left\{ f : \sup_{t>0} \frac{K_r(f, \tilde{\Delta}, t^{2r})_p}{t^\alpha} < \infty \right\} \equiv L_p^\alpha(S^{d-1}), \tag{7.1}$$

and the definition does not depend on r as long as $2r > \alpha$.

We can now state and prove the following result.

Theorem 7.1. *For* $1 < p < \infty$ *and* $0 < \theta \leqslant \frac{\pi}{2}$, $f \in L_p(S^{d-1})$ *implies* $S_\theta f \in L_p^{\alpha_p}(S^{d-1})$ *with* $\alpha_p = (d-2)|1 - \frac{1}{p}|$ *for* $p \leqslant 2$, $\alpha_p = (d-2)/p$ *for* $p \geqslant 2$ *and*

$$K_r(S_\theta f, \tilde{\Delta}, t^{2r})_p \leqslant C \theta^{-\alpha_p} t^{\alpha_p} \|f\|_p \tag{7.2}$$

with C *independent of* t *and* θ.

Proof. We set $\eta(x) \in C^\infty[0, \infty)$ such that $\eta(x) = \begin{cases} 1 & x \leqslant \frac{1}{2} \\ 0 & x \geqslant 1 \end{cases}$ and we denote $\eta_j(x) \equiv \eta(\frac{x}{2^j})$. For $f \in L_p(S^{d-1})$, $O_j f$ given by

$$O_j f \sim \sum \eta_j\left(\frac{k}{n}\right) P_k(f) \tag{7.3}$$

are multiplier operators which satisfy

$$\|O_j f\|_p \leqslant C\|f\|_p, \quad 1 \leqslant p \leqslant \infty, \quad j = 0, 1, \ldots \tag{A}$$

and

$$E_{2^j}(f)_p \leqslant \|f - O_j f\|_p \leqslant C E_{2^{j-1}}(f)_p, \quad 1 \leqslant p \leqslant \infty, \quad j = 1, 2, \ldots, \tag{B}$$

where

$$E_n(f)_p = \inf(\|f - P\|_p; \; P \text{ spherical harmonic of degree } \leqslant n).$$

In fact, in our proof, any bounded sequence of multiplier operators satisfying (A) and (B) can replace O_j. (For instance, any delayed means V_λ generated by some Riesz means as discussed in [Ch-Di].) We now set $T_j f \equiv O_j f - O_{j-1} f$ for $j \geqslant 1$ and $T_1 f = O_0 f$. We show now that for $1 \leqslant p \leqslant \infty$ and $2r > \beta$

$$C^{-1} \sup_{j \geqslant 1} \frac{\|T_j f\|_p}{2^{-j\beta}} \leqslant \sup_{t > 0} \frac{K_r(f, \tilde{\Delta}, t^{2r})_p}{t^\beta} \leqslant C \sup_{j \geqslant 1} \frac{\|T_j f\|_p}{2^{-j\beta}}. \tag{7.4}$$

The first inequality follows as $\|T_j f\|_p \leqslant A E_{2^{j-1}}(f)_p$ and $E_n(f)_p$ is bounded by $K_r(f, \tilde{\Delta}, n^{-2r})_p$ (see for instance [Ch-Di, (8.8)]). The second implication follows from

$$E_{2^j}(f)_p \leqslant \|f - O_j f\|_p \leqslant \sum_{\ell=j+1}^{\infty} \|T_\ell f\|_p \leqslant A 2^{-j\beta} \sup_\ell \left(\frac{\|T_\ell f\|_p}{2^{-\ell\beta}}\right),$$

with A depending on β (but not on j or f), and the estimate of $K_r(f, \tilde{\Delta}, t^{2r})_p$ by $E_n(f)_p$ (see for instance [Ch-Di, (8.9)]). We now observe that for $p = 1$

$$\|T_j(S_\theta f)\|_1 \leqslant \|O_j(S_\theta f)\|_1 + \|O_{j-1}(S_\theta f)\|_1 \leqslant 2C\|S_\theta f\|_1 \leqslant 2C\|f\|_1.$$

For $p = 2$ we use $\|T_j(S_\theta f)\|_2 \leqslant A E_{2^{j-1}}(S_\theta f)_2$, and as

$$|(\theta k)^{(d-2)/2} Q_k^{(\lambda)}(\cos\theta)| \leqslant C \quad \text{for all } k,$$

we have $\|(-\tilde{\Delta})^{(d-2)/4} S_\theta f\|_2 \leqslant 2C\theta^{-(d-2)/2}\|f\|_2$, and hence (see [Di-II, Theorem 4.1 and also A, p. 343]) $E_{2^j}(S_\theta f)_2 \leqslant C_1 \theta^{-(d-2)/2} 2^{-j(d-2)/2}\|f\|_2$.

As T_j and S_θ are linear operators, we may use the Riesz–Thorin theorem to obtain $\|T_j S_\theta f\|_p \leqslant C\theta^{-\alpha_p} 2^{-jd_p}\|f\|_p$ with $\alpha_p = (d-2)(1 - \frac{1}{p})$ for $1 < p < 2$. For $2 < q < \infty$ we get the same result by duality and $\alpha_q = \alpha_p$ for $\frac{1}{p} + \frac{1}{q} = 1$. □

We cannot show that for $1 < p < 2$ (or for $2 < p < \infty$) Theorem 7.1 is best possible and it probably is not. For $p = 2$ Theorem 7.1 is best possible and we will show below that for $p = 1$ (and hence by duality for $p = \infty$) Theorem 7.1 cannot be improved either.

Theorem 7.2. *There exists* $f \in L_1(S^{d-1})$ *such that* $(-\tilde{\Delta})^{\alpha} S_{\theta} f \notin L_1(S^{d-1})$ *for any positive* α.

Remark 7.3. We note that using Theorem 3.1, we established that for $m > \frac{2\left(\left[\frac{d}{2}\right]+3\right)}{d-2}$ $\tilde{\Delta} S_{\theta}^{m} f \in L_1(S^{d-1})$, and in particular if $d \geqslant 10$, we have $\tilde{\Delta} S_{\theta}^{2} f \in L_1(S^{d-1})$, which makes Theorem 7.2 somewhat intriguing.

First we state the Nikolskii inequality for spherical polynomials.

Lemma 7.4. *For* P_n, *a spherical polynomials of degree smaller or equal to* n *on* S^{d-1} (*that is,* $P_n \in span(\bigcup_{k=0}^{n} H_k)$ *where* $H_k = \{\psi : \tilde{\Delta}\psi = -k(k+d-2)\psi\}$), *and* $0 < q < p \leqslant \infty$ *we have*

$$\|P_n\|_p \leqslant C n^{(d-1)(\frac{1}{q}-\frac{1}{p})} \|P_n\|_q. \tag{7.5}$$

Proof. While we found a Ref. [Ka] for the part of the result which we need i.e. when $q \geqslant 1$, we give below a short proof of (7.5). We choose integer $r > \frac{q}{2}$. $P_n(x)^r$ is a spherical polynomials of degree nr. We write

$$P_n(x)^r = \int_{S^{d-1}} P_n(y)^r \sum_{k=0}^{nr} \sum_{\ell=1}^{d_k} Y_{k,\ell}(x) Y_{k,\ell}(y)\, dy = \int_{S^{d-1}} P_n(y)^r\, G_{nr}(x \cdot y)\, dy,$$

where $\{Y_{k,\ell}\}_{\ell=1}^{d_k}$ is any orthonormal system in H_k. We now have

$$\begin{aligned} \|P_n\|_{\infty}^{r} &\leqslant \|P_n\|_{\infty}^{r-q/2} \int_{S^{d-1}} |P_n(y)|^{q/2} |G_{nr}(x \cdot y)|\, dy \\ &\leqslant \|P_n\|_{\infty}^{r-q/2} \|P_n\|_{q}^{q/2} \left(\int (G_{nr}(x \cdot y))^2\, dy \right)^{1/2}. \end{aligned}$$

As $\int_{S^{d-1}} (G_{nr}(x \cdot y))^2 dy$ is independent of x, it is equal to $\frac{1}{|S^{d-1}|} \sum_{k=0}^{nr} \dim H_k \approx n^{d-1}$, and the proof of (7.5) when $p = \infty$ follows.

To complete the proof we recall for $q < p < \infty$, $\|f\|_p \leqslant \|f\|_q^{q/p} \|f\|_{\infty}^{1-q/p}$. □

Lemma 7.5. *If* $(-\tilde{\Delta})^{\alpha} f \in L_1$, $(\alpha < 1)$, *then* $f \in L_p$ *for* $1 \leqslant p < p(\alpha) = \frac{d-1}{d-1-2\alpha}$.

Proof. Recall T_j from the proof of Theorem 7.1, and following [Di-II, Theorem 4.1], $\|T_j\|_1 \leqslant C E_{2^{j-1}}(f)_1 \leqslant C_1 2^{-2j\alpha}$. Using Lemma 7.4 with $q = 1$, $\|T_j\|_p \leqslant C_2 2^{-2j\alpha} (2^j)^{(d-1)(1-\frac{1}{p})}$, and if $2\alpha > (d-1)(1-\frac{1}{p})$ or $p < \frac{d-1}{d-1-2\alpha}$,

$\|\sum_{j=1}^{\infty} T_j\|_p \leqslant \sum \|T_j\|_p < \infty$. As $\|T_0\|_p \leqslant C\|f\|_1$ and $\sum_{j=0}^{\infty} T_j = f$, we complete the proof. $\square$

Proof of Theorem 7.2. In view of Lemma 7.5 it is sufficient to construct a function $f \in L_1(S^{d-1})$ such that $S_\theta f \notin L_p(S^{d-1})$ for any $p > 1$. Our strategy is to use $f_n(x) \geqslant 0$ so that $S_\theta f_n \geqslant 0$. We then set $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} f_n(x)$, which implies $\|S_\theta f\|_p \geqslant \frac{1}{n^2}\|S_\theta f_n\|_p$, and show that $\sup_n \frac{1}{n^2}\|S_\theta f_n\|_p = \infty$ for any $p > 1$. To do that we write

$$f_n(x) = 2^{n(d-1)}\psi_{2^{-n}}(x \cdot z_n), \quad \psi_\delta(t) = \begin{cases} 1 & t \geqslant \cos\delta, \\ 0 & t < \cos\delta. \end{cases} \tag{7.6}$$

In fact, z_n are immaterial and we can choose $z_n = z$ if we wish. Clearly $\|f_n\|_1 \leqslant C$ and $f \in L_1(S^{d-1})$ while $\|f_n\|_p \approx 2^{n(d-1)(1-\frac{1}{p})}$ and $f \notin L_p(S^{d-1})$ for $p > 1$. We will now show that for $\theta \gg \eta > 0$, for example $\theta > 10\eta > 0$ (but as η takes the role of 2^{-n}, we could have a much larger ratio)

$$\begin{aligned} &m(y : y \cdot x = \cos\theta, y \cdot z \geqslant \cos\eta) \geqslant C\eta^{d-2}, \\ &\quad \text{if} \quad \cos\left(\theta + \tfrac{\eta}{2}\right) \leqslant x \cdot z \leqslant \cos\left(\theta - \tfrac{\eta}{2}\right) \end{aligned} \tag{7.7}$$

with C independent of η. As $m\{x : \cos(\theta + \frac{\eta}{2}) \leqslant x \cdot z \leqslant \cos(\theta - \frac{\eta}{2})\} \approx \theta^{d-2}\eta$ the demonstration of (7.7) will complete the proof of our theorem setting $\eta = 2^{-n}$ (and any fixed θ) and obtaining

$$\frac{1}{n^2}\|S_\theta f_n\|_p \approx \frac{1}{n^2}([(2^{-n})^{(d-2)}2^{n(d-1)}]^p \theta^{d-2}2^{-n})^{1/p} \approx \frac{\theta^{(d-2)/p}}{n^2} 2^{n(1-\frac{1}{p})}.$$

Hence, it remains only to prove (7.7). We observe that if $\cos(\theta - \frac{\eta}{2}) \leqslant x \cdot z \leqslant \cos(\theta - \frac{\eta}{2})$, then there exists z_0 such that $z_0 \cdot z \geqslant \cos\frac{\eta}{2}$ and $x \cdot z_0 = \cos\theta$. This follows since in the plane containing the vectors x and z, the angle between them has to be increased or decreased by less than $\frac{\eta}{2}$ to get z_0. The measure $m(y : y \cdot x = \cos\theta, y \cdot z_0 \geqslant \cos\frac{\eta}{2}, z_0 \cdot x = \cos\theta)$ can be calculated to behave like η^{d-2}. Our result follows as

$$\begin{aligned} &\left\{y : y \cdot x = z_0 \cdot x = \cos\theta, y \cdot z_0 \geqslant \cos\tfrac{\eta}{2}\right\} \\ &\quad \subset \left\{y : y \cdot x = \cos\theta, y \cdot z_0 \geqslant \cos\tfrac{\eta}{2}, \cos\left(\theta + \tfrac{\eta}{2}\right) \leqslant z_0 \cdot x \leqslant \cos\left(\theta - \tfrac{\eta}{2}\right)\right\} \\ &\quad \subset \{y : y \cdot x = \cos\theta, y \cdot z \geqslant \cos\eta, \cos(\theta + \eta \leqslant x \cdot z \leqslant \cos(\theta - \eta)\}. \quad \square \end{aligned}$$

We also have the (slightly simpler to prove) analogues in R^d (or T^d).

Theorem 7.6. *For* $1 < p < \infty$, $f \in L_p$ ($L_p(T^d)$ or $L_p(R^d)$) *implies* $V_t f \in L_p^{\alpha_p}$ *with* $\alpha_p = (d-1)|1 - \frac{1}{p}|$ *if* $p \leqslant 2$ *and* $\alpha_p = (d-1)/p$ *for* $p \geqslant 2$ *where*

$$L_p^\alpha \equiv \left\{f ; \sup_{u>0} \frac{K_r(f, \Delta, u^{2r})_p}{u^\alpha} < \infty\right\}, \quad 2r > \alpha$$

and

$$K_r(f,\Delta,u^{2r})_p \equiv \inf_g(\|f-g\|_p + u^{2r}\|\Delta^r g\|_p).$$

Moreover,

$$K_r(V_t f,\Delta,u^{2r})_p \leqslant Ct^{-\alpha_p}u^{\alpha_p}\|f\|_p. \tag{7.8}$$

Theorem 7.7. *There exists* $f\in L_1$ *such that* $(-\Delta)^\alpha V_t f\notin L_1$ *for any positive* α.

Remark 7.8. Theorems 7.6 and 7.7 are similar and perhaps easier to prove than Theorems 7.1 and 7.2. We may prove these theorems first for T^d and deduce them for R^d or vice versa. Analogues of Lemmas 7.4 and 7.5 for T^d and trigonometric polynomials of deg n are easy to prove and perhaps better known than Lemmas 7.4 and 7.5. We did not give the proofs explicitly to avoid duplication.

Remark 7.9. In Theorems 7.1 and 7.2 the space L_p^α appears to characterize smoothness. The same situation occurs if we use the new modulus of smoothness introduced in [Di-I], that is

$$\omega^m(f,t)_p = \sup\left\{\left\|\Delta_\rho^m f\right\|_p;\ \rho x\cdot x\geqslant \cos t,\ \rho\in \mathrm{SO}(d)\right\} \tag{7.9}$$

or if we use best spherical harmonic approximation. The above follows from the fact that for $1\leqslant p\leqslant\infty$

$$\omega^m(f,t)_p = O(t^\alpha) \quad \text{for } m>\alpha,$$

$$K_r(f,\tilde{\Delta},t^{2r})_p = O(t^\alpha) \quad \text{for } 2r>\alpha$$

and

$$E_n(f)_p = O\left(\frac{1}{n^\alpha}\right)$$

are all equivalent (see [Di-III]).

It is in some more sensitive conditions that the concepts $E_n(f)_p$, $K_r(f,\tilde{\Delta},t^{2r})_p$ and $\omega^m(f,t)_p$ may differ.

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